Multiplicative thinking

Why is multiplicative thinking an important notion?

Multiplicative thinking has been identified as one of the most important and also most difficult mathematical concepts for learners to develop (Siemon, Virgona & Cornielle, 2001). Multiplicative thinking is a prerequisite for the understanding of rational number and proportional reasoning: fractions, decimals, ratio, percent and proportion (Siemon, Izard, Breed & Virgona, 2006). Successful transition from additive to multiplicative thinking is necessary before learners can fully understand proportional reasoning and algebra in high school mathematics.

This paper provides guidelines on developing multiplicative thinking and how leaders can support educators in fostering this development, both in themselves and in their learners.

What is multiplicative thinking?

Multiplicative thinking incorporates the capacity to work flexibly with the concepts, strategies and representations of multiplication (and division) as they occur in a wide range of contexts, together with the ability to communicate this in words, diagrams or symbolic expressions.

Multiplicative thinking involves recognising and working with *relationships between* numbers whereas additive thinking only requires a capacity to work with numbers themselves. While some simple 'multiplication problems' can indeed be solved with additive thinking, difficulties will arise when moving onto larger whole numbers, and fractions and decimals. So learners need a well-developed sense of number, as well as experience with the many different contexts in which multiplication and division arise.

Developing multiplicative thinking

Multiplicative thinking develops from two preparatory activities: counting and splitting.

Counting-based ideas build on young children's experience with addition and subtraction. These include:

- counting large collections (to encourage counting using 2s and 5s for efficiency)
- repeated addition
- skip counting
- repeated subtraction.

These strategies are often used when working with a range of multiplicative problems that involve relatively small whole numbers. They do not support the thinking required when generalising with larger numbers and working with fractions and decimals. The transition from additive strategies to meaningful mental strategies that support multiplicative thinking more generally requires a significant shift in thinking away from *a count of equal groups* and *repeated addition* to a *constant* number of groups. For instance thinking in terms of a *count of equal groups* to solve 3 multiplied by 8, a student may skip count by three, eight times (3, 6, 9, 12, 15, 18, 21, 24). In comparison, thinking in terms of a constant number of groups to solve 3 multiplied by 8, as student may think double eight (16) and one more eight, 24. When learners are observed using counting-based ideas to solve multiplicative problems, educators can use questioning and enabling prompts which incorporate language such as *for each* and *times as many* to promote multiplicative thinking.

The *splitting* notion involves enlarging, shrinking and replicating: for example, doubling rather than adding on the same amount again. This notion helps learners

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to know not only what would be in three bags each containing four marbles, but also what would be in five bags, ten bags, twenty bags etc. This is particularly useful when working with fractions and decimals. For instance, when renaming halves as quarters, this involves recognition that for each half, there are two smaller parts.

How can educators help learners to develop multiplicative thinking?

The following notions are developmental but not strictly linear.

There is often overlap as learners learn to:

- **count in groups, as long as all of the groups are visible**
- **manage situations involving multiplication and division by holding both the number of objects within each group and the number of groups in their head at the same time (Jacob & Willis, 2001)**
- **make and name arrays and regions, eg 3 fours which can be rotated and renamed as 4 threes. Learners then start to use commutativity, inverse relations, and part-whole understanding (Mulligan & Watson, 1998).**
- **develop Cartesian product representations (such as tree diagrams) and manipulation. This relates to all the possible options when combining elements from two different sets. For example, if I have four shirts and three skirts then there are 12 possible shirt/skirt combinations. Furthermore, if I have four shirts, three skirts and two jumpers then there are 24 possible shirt/skirt/jumper combinations. The thinking here, is that *for each* shirt, there are 3 types of skirt, and *for each* skirt, there are two types of jumper. The resulting set is the Cartesian product.**

To become confident multiplicative thinkers it is necessary for learners to simultaneously recognise and coordinate the number of groups (multiplier) and the number in each group (multiplicand). So, three bags of four marbles are different from four bags of three marbles, not in terms of how many marbles altogether, but in terms of how many groups (number of bags) and how much in each group (the number in each bag).

Using arrays, regions and area models to teach multiplication has been shown to help learners develop conceptual understanding of multiplicative thinking in a relatively short time frame (Norbury, 2002). Arrays help learners to see both *factors* (multiplicand and multiplier) and the *product* at the same time when used in conjunction with questioning and enabling prompts to extend learner understanding of the *for each* and *times as many* ideas.

How can leaders support their staff?

Creating collaborative structures and opportunities, through which educators explore a range of contexts and multiple representations for multiplication problems, promotes the development of multiplicative thinking. Leaders support enhanced practice when they provide opportunities for educators to engage with and trial high quality models, using professional learning and research, to design learning experiences that focus on the language of multiplicative thinking, for example *for each* and *times as many*.

Building pedagogical and content knowledge around the developmental ideas for multiplicative thinking is important to enable educators to recognise and address learner difficulties with multiplication and division. For example, knowing that a learner can count in groups, but has difficulty holding the number of groups in her head, enables educators to intentionally design problems that draw the learner's attention to the number of groups.

When observing lessons, look for learners who:

- use the language of multiplicative thinking (eg *for each* and *times as many*)
- use efficient strategies based on working with arrays and regions (eg 3 'of anything', is 'double it and one more of it')
- can work flexibly with concepts, strategies and representations of multiplication (and division) in a wide range of contexts
- use a variety of representations, including words, arrays, regions and area models to show multiplicative thinking when problem solving.

Leaders should consider how they work with their staff to incorporate the big ideas in number into common agreements around planning, teaching and assessment at their site.

Reflective questions for leaders to ask their teachers

When looking at and discussing the numeracy and mathematics program, you could, for example, ask the teacher:

- Can students recognise when a problem requires them to use multiplicative thinking? Are they able to work meaningfully with the relationships between numbers?
- For those who seem to be struggling to understand how multiplicative thinking can be used (counting all, skip counting, and repeat addition), what steps are you taking to address this learning need?
- Have you used the multiplicative thinking common misunderstanding tools? (Siemon, 2009)
- What kind of problems are you asking students to solve that you think are appropriate to their stage of development?
- What opportunities do you provide for your students to deepen their understanding of multiplicative thinking through making links to their daily life?

Further resources

The big ideas in number are discussed in further detail in the following mathematics papers:

- 3.0 Conceptual understanding: Number and algebra
- 3.1 Trusting the count
- 3.2 Place value
- 3.4 Partitioning
- 3.5 Proportional reasoning
- 3.6 Generalising.

All papers in this series are based on the work of Dianne Siemon, Professor of Mathematics Education at RMIT and a key text (Siemon et al, 2015).

http://bit.ly/BestAdviceSeries

Further reading

ACER PAT Teaching Resources Centre houses relevant examples of multiplicative thinking, for example:

- multiplication strategies
- percentage of a quantity
- finding factors and multiples
- commutative law and multiplication.

Jacob L & Willis S (2003) 'The development of multiplicative thinking in young children', Deakin University. Available at http://researchrepository.murdoch.edu.au/6178/ (accessed January 2017). This paper describes developmental changes as children move from additive to multiplicative thinking.

Van De Walle JA, Karp K & Bay-Williams JM (2016) *Elementary and Middle School Mathematics: Teaching Developmentally*, Ninth global edition, UK: Pearson Education Limited. In particular, refer to chapter on 'Algebraic thinking: Generalisations, patterns and functions'.

Victorian Department of Education and Training, Mathematics Developmental Continuum F–10 This resource provides evidence-based indicators of progress, linked to powerful teaching strategies.

Victorian Department of Education and Training, Assessment for Common Misunderstandings These tools draw on highly focussed, research-based Probe Tasks and the Probe Task Manual (RMIT), as well as a number of additional tasks and resources which have been organised to address 'common misunderstandings'.

References

Jacob L & Willis S (2001) 'Recognising the difference between additive and multiplicative thinking in young children', paper presented at the *Numeracy and Beyond: Proceedings of the 24th Annual Mathematics Education Research Group of Australasia Conference*, Sydney: MERGA. Available from https://www.merga.net.au/documents/RR_Jacob&Willis.pdf

Mulligan J & Watson J (1998) 'A developmental multimodal model for multiplication and division', *Mathematics Education Research Journal*, 10(2), 61–86

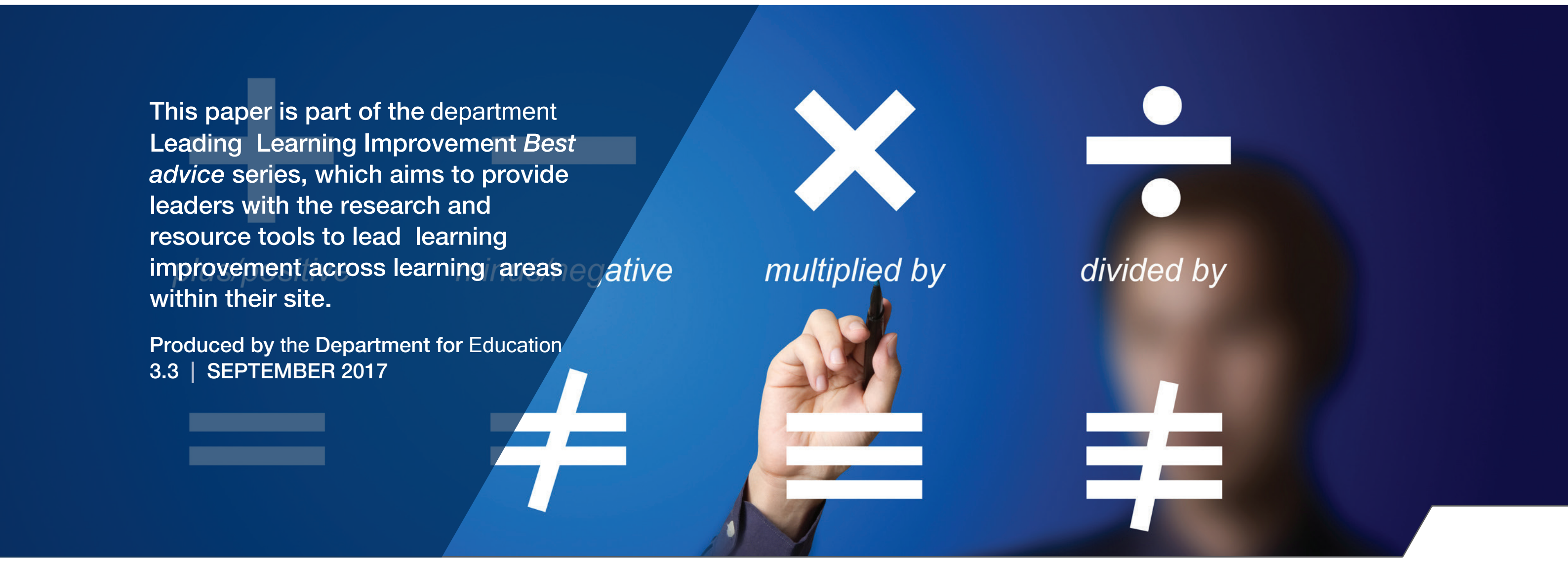
Norbury H (2002) 'Models and representations: Do they have a role in a conceptual understanding of multiplication?' Paper presented at the *Mathematics – making waves: Proceedings of the Nineteenth Biennial Conference of The Australian Association of Mathematics Teachers Inc*, The University of Queensland, Brisbane. Available at www.aamt.edu.au/content/download/19062/252031/file/m~waves.pdf

Siemon D (2009) *Multiplicative thinking tools: Common misunderstanding*, Department of Education and Early Childhood Development, State of Victoria, available from https://edi.sa.edu.au/library/document-library/learning-improvement/strategic-design/di-siemon-diagnostic-tools/3-MULTIPLICATIVE-THINKING-DIAGNOSTIC.pdf

Siemon D, Beswick K, Brady K, Clark J, Faragher R & Warren E (2015) *Teaching Mathematics: Foundations to Middle Years*, 2nd edition, Melbourne, Oxford University Press

Siemon D, Izard J, Breed M & Virgona J (2006) 'The derivation of a learning assessment framework for multiplicative thinking', in J Novotna, H Moraova, M Kratka & N Stehlikova (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, Prague*, Czech Republic: PME, (pp. 113–120). Available at http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.492.3231 (accessed January 2017). This paper and the supporting files found on the Victorian Education website (above) give simple and clear examples of diagnostic tasks for multiplicative thinking.

Siemon D, Virgona J & Cornielle K (2001) 'The middle years numeracy research project: 5–9', Victoria: RMIT, available at www.eduweb.vic.gov.au/edulibrary/public/curricman/middleyear/mynumeracyresearchfullreport.pdf (accessed January 2017)



This paper is part of the department Leading Learning Improvement *Best advice* series, which aims to provide leaders with the research and resource tools to lead learning improvement across learning areas within their site.

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